Introduction to Quantum Noise

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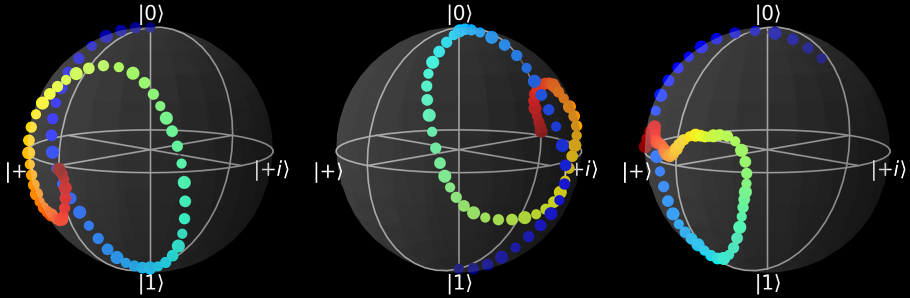
Previously



How can we combine these possibilities?

Previously

Suppose we have some probabilities for each initial state:

- 90%: $|0\rangle$
- 8%: $|1\rangle$
- 2%: $|\psi_r\rangle = \frac{1}{\sqrt{10}}(3i|0\rangle - |1\rangle)$

It would be nice to have some mathematical expression to express the possible measurement outcomes!

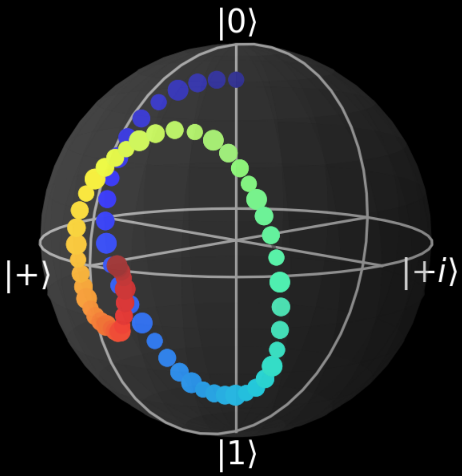
Previously: Density States

$$\rho = \sum_i p_i \left|\psi\right\rangle \left\langle\psi\right|$$

$$\rho = 0.9 \left|0\right\rangle \left\langle 0\right| + 0.08 \left|1\right\rangle \left\langle 1\right| + 0.02 \left|\psi_r\right\rangle \left\langle\psi_r\right|$$

Previously: Density States

$$\rho' = H\rho H^\dagger$$



Previously: Errors

1. Coherent Errors
2. Incoherent Errors
3. Decoherent Errors

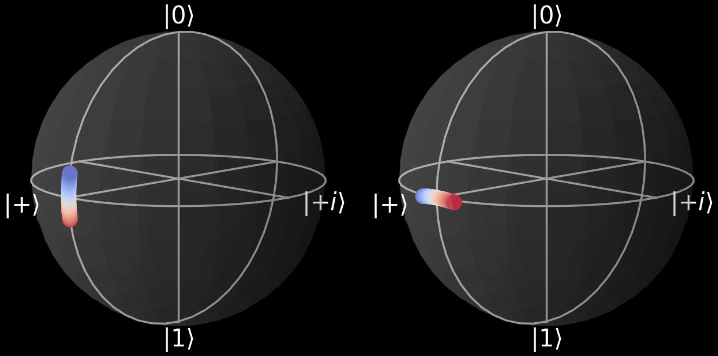
Quantum Operations

$$\mathcal{E}(\rho) \rightarrow \rho'$$

Example

Suppose we have two options:

- 30%: $U(\delta)$
- 50%: $U(\gamma)$



Example

What is the average effect?

$$\mathcal{E}(\rho) = \int p(\delta)d\delta U(\delta)\rho U^\dagger(\delta) + \int p(\gamma)d\gamma U(\gamma)\rho U^\dagger(\gamma)$$

$$\mathcal{E}(\rho) = 0.3U(\delta)\rho U^\dagger(\delta) + 0.5U(\gamma)\rho U^\dagger(\gamma)$$

Some Popular Operations

Identity:

$$\mathcal{E}(\rho) = \rho$$

Fully depolarizing:

$$\mathcal{E}(\rho) = \mathbf{tr}(\rho)\,\frac{1}{d}\mathbb{I}$$

Dephasing:

$$\mathcal{E}(\rho) = \gamma\rho + (1-\gamma)Z\rho Z$$

Today's Goal

We want to estimate how "close" a quantum operation $\mathcal{E}$ is to our desired unitary operator U.

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1. $\mathcal{E}$ will not be perfect!
2. What about different states!?

Average Gate Fidelity

What is the fidelity on average?

$$\bar{F}(\mathcal{E}, V) = \int_{\mathcal{U}} d\mu(U_r) \langle 0 | U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle\langle 0| U_r^\dagger) U U_r |0\rangle .$$

Today's goal: to understand what this means and how to compute $\bar{F}$

Overview

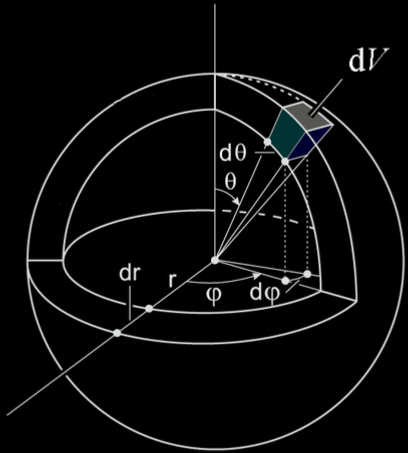
1. Haar measure and random states
2. Unitary t-designs
3. Finale: obtain $\bar{F}$

Haar Measure and Random States

$$\bar{F}(\mathcal{E}, V) = \int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle\langle 0| U_r^\dagger) U U_r |0\rangle .$$

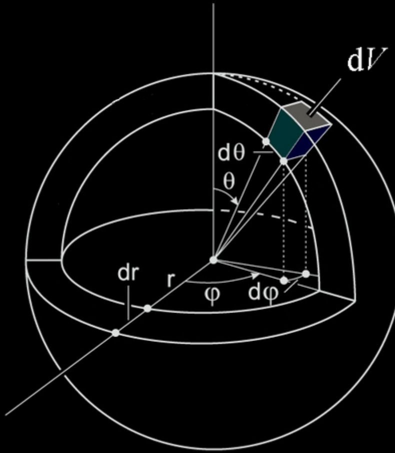
What do $d\mu(U_r)$, $\mathcal{U}$, $U_r |0\rangle$ mean?

What is a *Measure*?



What is a *Measure*?

$$V = \int_0^R \int_0^{2\pi} \int_0^{\pi} dr\, d\phi\, d\theta = 2\pi^2 R$$



What is a *Measure*?

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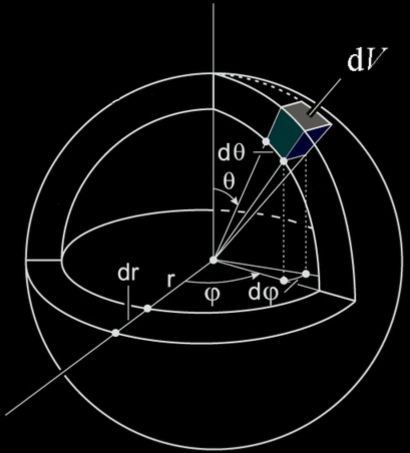
$$V = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta\, dr\, d\phi\, d\theta = \frac{4}{3}\pi R^3$$



What is a *Measure*?

The *measure* tells us how things are distributed and concentrated in a mathematical space.

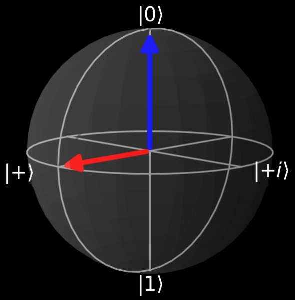
Example: Qubit Unitary

$$U(\phi,\theta,\omega) = \begin{pmatrix} e^{-i(\phi+\omega)/2}\cos(\theta/2) & -e^{i(\phi-\omega)/2}\sin(\theta/2) \\ e^{-i(\phi-\omega)/2}\sin(\theta/2) & e^{i(\phi+\omega)/2}\cos(\theta/2) \end{pmatrix}$$

$$\phi = 0$$

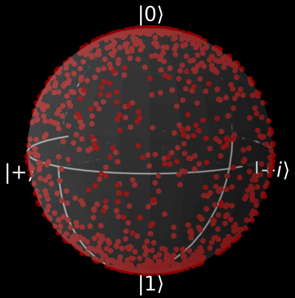
$$\theta = \pi/2$$

$$\omega = 0$$



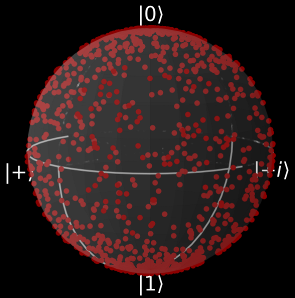
Example: Qubit Unitary

$$\phi, \theta, \omega \sim \textbf{uniform}(0, 2\pi)$$



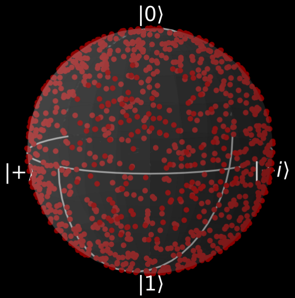
Example: Qubit Unitary

$$\phi, \theta, \omega \sim \mathrm{uniform}(0, 2\pi)$$



$$\phi, \omega \sim \mathrm{uniform}(0, 2\pi)$$

$$\theta \sim D(0, 2\pi);\ \Pr(\theta) = \sin\theta$$



Haar Measure

To integrate over unitary operators we must use the *Haar measure*!

$$\int_{V \in \mathcal{U}(N)} f(V) d\mu_N(V)$$

$$d\mu_2 = \text{sin}\,\theta d\theta \cdot d\omega \cdot d\phi$$

Haar Measure and Random States

$$\bar{F}(\mathcal{E}, V) = \int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle\langle 0| U_r^\dagger) U U_r |0\rangle .$$

What do $d\mu(U_r)$, $\mathcal{U}$, $U_r |0\rangle$ mean?

Haar Measure and Random States

$$\bar{F}(\mathcal{E}, V) = \int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle\langle 0| U_r^\dagger) U U_r |0\rangle.$$

- $d\mu(U_r)$ is the Haar measure
- $\mathcal{U}$ is the space of Unitary operators
- $U_r |0\rangle$ is a random state $|\psi_r\rangle$

Haar Measure and Random States

$$\bar{F}(\mathcal{E}, V) = \int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle \langle 0| U_r^\dagger) U U_r |0\rangle \,.$$

1. Prepare a random state $|\psi_r\rangle = U_r |0\rangle$
2. Compare $\mathcal{E}(|\psi_r\rangle \langle \psi_r|)$ and $U |\psi_r\rangle \langle \psi_r| U^\dagger$
3. Repeat and compute average

Haar Measure and Random States

Drawbacks:

1. Picking a random unitary is exponentially hard with dimension N
2. Need to execute $\mathcal{E}$ many times to compute average

Can we do better?

Overview

1. ~~Haar measure and random states~~
2. Unitary t-designs
3. Finale: obtain $\bar{F}$

What is a t-design?

Suppose we have a polynomial of d-variables, and we seek to compute the average of the polynomial over a d-dimensional unit sphere

Example: $f(x, y, z) = x^4 - 4x^3y + y^2z^2$

$$\int f(x, y, z) d\mu(x, y, z) = 4/15 \approx 0.266667$$

What is a t-design?

1. Integrate the polynomial over the sphere, using a proper measure
2. Approximate the average value by sampling points of the sphere uniformly, computing the function value at those points, and then average them

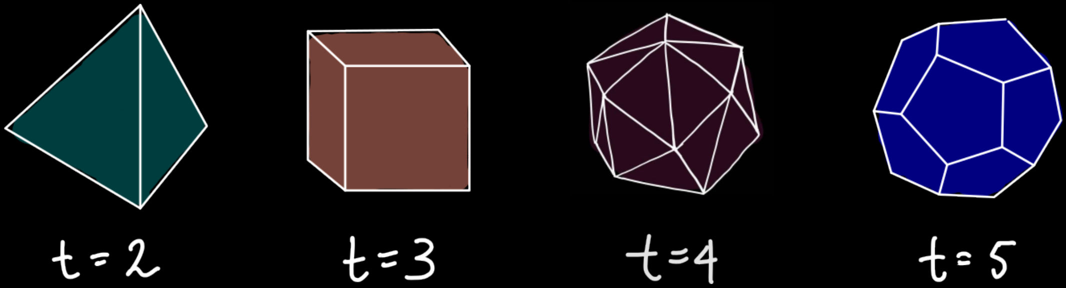
What is a t-design?

Theorem (t-designs)

If the terms in a polynomial all have the same degree of at most t, then there are a set of points X that will give the average of the polynomial exactly.

$$\frac{1}{|X|}\sum_{x\in X} p_t(x) = \int p_t(u)d\mu(u)$$

Example: Spherical t-design



Example: Spherical t-design

$$f(x, y, z) = x^4 - 4x^3y + y^2z^2$$

All terms have degree 4, so $t = 4$.

Example: Spherical t-design

```
def f(x, y, z):
    return (x ** 4) - 4 * (x ** 3) * y +  y ** 2 * z ** 2

dodeca_average = np.mean([f(*point) for point in dodecahedron])
print(dodeca_average)
```

```
0.26666666666666668
```

Unitary t-design

$$\frac{1}{K}\sum_{k=1}^{K} P_{t,t}(U_k) = \int_{\mathcal{U}(d)} P_{t,t}(U)d\mu(U)$$

Unitary t-design

$$\frac{1}{K}\sum_{j=1}^{K} \langle 0| U_{rj}^{\dagger} U^{\dagger} \mathcal{E}(U_{rj} |0\rangle \langle 0| U_{rj}^{\dagger}) U U_{rj} |0\rangle$$

$$= \int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^{\dagger} U^{\dagger} \mathcal{E}(U_r |0\rangle \langle 0| U_r^{\dagger}) U U_r |0\rangle$$

Unitary t-design

What are the representative set of points?

Unitary t-design

- the Pauli group $P = \{I, X, Y, Z\}$ form a unitary 1-design
- the Clifford group forms a unitary 3-design (also 2-design and 1-design)

Clifford group

$$CPC^\dagger = \pm P', \quad \forall P, P' \in \mathcal{P}(n), \quad C \in \mathcal{C}(n).$$

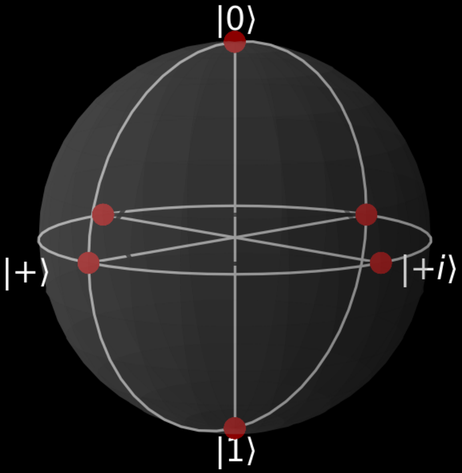
Example

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

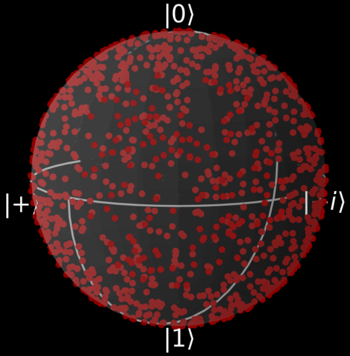
Qubit Clifford group:

```
'I',
'H', 'S',
'HS', 'SH', 'SS',
'HSH', 'HSS', 'SHS', 'SSH', 'SSS',
'HSHS', 'HSSH', 'HSSS', 'SHSS', 'SSHS',
'HSHSS', 'HSSHS', 'SHSSH', 'SHSSS', 'SSHSS',
'HSHSSH', 'HSHSSS', 'HSSHSS'
```

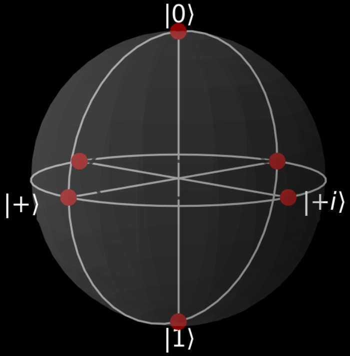


Average Gate Fidelity

$$\int_{\mathcal{U}} d\mu(U_r) \langle 0| U_r^\dagger U^\dagger \mathcal{E}(U_r |0\rangle \langle 0| U_r^\dagger) U U_r |0\rangle$$



$$\frac{1}{K} \sum_{j=1}^{K} \langle 0| U_{rj}^\dagger U^\dagger \mathcal{E}(U_{rj} |0\rangle \langle 0| U_{rj}^\dagger) U U_{rj} |0\rangle$$



Conclusion

Summary:

- Haar measure
- Random quantum states
- t-designs
- Average Gate Fidelity

Not discussed:

- Quantum process tomography ($\mathcal{E}$)
- Approximate t-designs